PMM U.S.S.R.,Vol.46,pp. 543-549
0021-8928/83/4 0543 $7.50/0

UDC 539.376

STRENGTH OPTIMIZATION OF THE SHAPE OF A VISCOELASTIC INHOMOGENEOUSLY-AGING REINFORCED ROD*

L.V. GENKIN and V.B. KOLMANOVSKII

The problem of selecting the shape of a minimal volume rod for which normal stresses in the reinforcements and the main material do not exceed given values is examined. Relationships are established that govern the optimal shape. The dependence of the optimal shape on the magnitude of the bending moment and functions characterizing the inhomogeneity of the aging is studied. Results are presented for numerical computations. The research continues the investigations in /1 − 3/.

1. Formulation of the problem. A viscoelastic reinforced rod of length l is in the undeformed state along the x-axis. A bending moment $M(x)$ acts on the rod in the longitudinal plane, which is its plane of symmetry. We introduce a Cartesian coordinate system y, z, in the rod cross-section, whose origin coincides with the center of gravity of this section. We take the neutral axis as the y axis, and the z axis perpendicular to y and in the plane of the transverse section (Fig.1). Let the subscripts a and 0 refer to the reinforcements and the main material. Thus $\sigma_a^i(t, x)$ is the normal stress in the i-th bar of the armature in the section $x \in [0, l]$ at the time $t \geqslant 0$; $i = 1, 2, \ldots, n$ (n is the number of bars in the reinforcement. Analogously, the normal stresses in the main material are denoted by $\sigma_0(t, x, z)$, where z is the distance to the neutral axis. The cross-sectional area $S(x)$ should be selected such that the volume of the rod

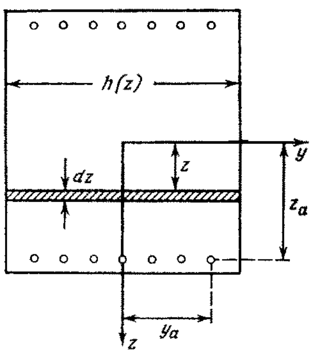

Fig.1

$$V = \int_0^l S(x)\,dx$$

would be minimal and constraints on the normal stresses would be satisfied (the numbers α_a and α_0 are given)

$$|\sigma_a^i(t, x)| \leqslant \alpha_a, \ |\sigma_0(t, x, z)| \leqslant \alpha_0 \qquad (1.1)$$
$$0 \leqslant x \leqslant l; \quad 0 \leqslant t; \quad i = 1, 2, \ldots, n$$

The reinforcement location is given and is fixed, where the projections of the reinforcement bars on the yz plane are enclosed in the rectangle $|y| \leqslant y_a$, $|z| \leqslant z_a$. The quantities y_a and z_a are given and identical for all sections.

Let us derive formulas for the normal stresses under the following assumptions: The center of gravity of the reinforcement agrees with the center of gravity of the main material in each section of the rod; the rod transverse sections remain planar during deformation and the law of plane sections is valid; the deformations $\varepsilon_a^i(t, x)$ and the stresses $\sigma_a^i(t, x)$ in the reinforcement are related by Hooke's law.

$$\sigma_a^i(t, x) = E_a \varepsilon_a^i(t, x) \qquad (1.2)$$

The main material is assumed viscoelastic and inhomogeneously aging, i.e., /1/

$$\varepsilon_0(t, x, z) = \sigma_0(t, x, z) E_0^{-1} - \int_0^t \sigma_0(\tau, x, z) \frac{\partial}{\partial \tau} C(t + \rho(x), \tau + \rho(x))\,d\tau \qquad (1.3)$$
$$C(t, \tau) = \varphi(\tau)[1 - e^{-\gamma(t-\tau)}], \ \gamma > 0, \ t \geqslant \tau$$

Here E_a and E_0 in (1.2) and (1.3) are constant elastic moduli, the piecewise continuous bounded function $\rho(x)$ is the age of an element with the coordinate x relative to an element with the coordinate $x = 0$; $C(t, \tau)$ is a measure of the creep, the function of the aging $\varphi(\tau) > 0$ is continuous, decreases monotonically, and tends to the measure of the creep C_0 of the main material in its old age as $\tau \to \infty$.

*Prikl.Matem.Mekhan.,46,No.4,683-690,1982

In conformity with the law of plane sections and (1.2), we have

$$\varepsilon_0(t, x, z) = \omega(t, x) z, \ \sigma_a^i(t, x) = E_a \omega(t, x) z_i \tag{1.4}$$

(the function $\omega(t, x)$ is the curvature of the rod cambered axis, and z_i is the distance between the y axis and the i-th bar of the reinforcement in the section x).

Let $R(E, t, \tau)$ be the resolvent of the kernel $(\partial/\partial\tau) C(t, \tau)$ with the parameter E. Then there results from (1.3) and (1.4) that

$$\sigma_0(t, x, z) = zb(t, x) \tag{1.5}$$

$$b(t,x) = E_0\Big(\omega(t,x) + E_0\int_0^t \omega(\tau, x) R(E_0, \tau + \rho(x), t + \rho(x))\, d\tau\Big)$$

Let us form the equilibrium equation

$$\sum_{i=1}^{n} S_a^i \sigma_a^i(t, x) z_i + \int_{S_0} \sigma_0(t, x, z) z h(z)\, dz = M(x)$$

Here S_a^i is the cross-sectional area of the i-th bar of the reinforcement, S_0 is the transverse section of the main material, and $h(z)$ is the width of a cross-sectional element. We replace σ_a^i and σ_0 by (1.4) and (1.5) in the equilibrium equations. Using I_a and I_0 for the corresponding moments of inertia of the sections with respect to the neutral axis, we obtain that

$$\omega(t, x) + (1-\beta) E_0 \int_0^t \omega(\tau, x) R(E_0, t + \rho(x), \tau + \rho(x))\, d\tau =$$

$$\frac{M(x)\beta}{I_a E_a}, \quad I_a = \sum_{i=1}^{n} S_a^i z_i^2, \quad I_0 = \int_{S_0} z^2 h(z)\, dz$$

$$\beta = I_a E_a (I_0 E_0 + I_a E_a)^{-1}, \quad 0 \leqslant \beta \leqslant 1$$

Solving the last integral equation for the curvature ω we conclude on the basis of the second relationship in (1.4) that the normal stress in the reinforcement is given by the formula

$$\sigma_a^i(t, x) = \frac{M(x)\beta}{I_a} z_i \Big(1 - (1-\beta) E_0 \times \int_0^t R(E_0\beta, t + \rho(x), \tau + \rho(x))\, d\tau\Big) \tag{1.6}$$

(the fact is used that the resolvent of the kernel $R(E, t, \tau)$ with the parameter $(1-\beta)E$ equals $R(E\beta, t, \tau)$ (see /4/, for instance)). Hence it is seen that the greatest normal stress will be in that bar of the reinforcement for which the quantity $|z_i|$ is maximal, i.e., equals z_a.

The representation

$$R(E, t, \tau) = \gamma \frac{\partial}{\partial\tau}\Big[\varphi(\tau) e^{\eta(\tau)} \int_\tau^t e^{-\eta(s)}\, ds\Big], \qquad \eta(s) = \gamma s + \gamma E \int_0^s \varphi(s_1)\, ds_1$$

is known from /1/. Consequently

$$\int_0^t R(E_0\beta, t + \rho(x), \tau + \rho(x))\, d\tau = J_0(t, \beta) = -\gamma\varphi(\rho(x)) \int_0^t \exp\Big[-\gamma s - E_0\beta\gamma \int_0^s \varphi(s_1 + \rho(x))\, ds_1\Big] ds \tag{1.7}$$

It is clear that the function $J_0(t, \beta)$ is negative and decreases monotonically in t. Hence, the relationship (1.6) permits rewriting the first of the inequalities (1.1) in the following equivalent form

$$F(\beta, \rho) \leqslant \frac{\alpha_a I_a}{M(x) z_a} = \bar{\alpha}_a \tag{1.8}$$

$$F(\beta, \rho) = \beta + \beta(1-\beta) J(\beta, \rho), \ J(\beta, \rho) = -E_0 \lim_{t\to\infty} J_0(t, \beta)$$

Let us manipulate the second of the constraints (1.1) in an analogous way. We first find an expression for the function $b(t, x)$ defined by (1.5). Substituting the functions (1.5) and (1.6) in the equilibrium equation and using the definition of the quantity β, we obtain

$$b(t, x) = M(x) E_0 E_a^{-1} I_a^{-1} \beta [1 + \beta E_0 J_0(t, \beta)] \tag{1.9}$$

Because of (1.5), (1.9), (1.7) and the function $J_0(t, \beta)$ that decreases monotonically in t, the second inequality in (1.1) is equivalent to the following ($r(x)$ is the distance from the y axis to the most remote point of the main material in the section x)

$$\beta r(x) \leqslant \bar{\alpha}_0, \quad \bar{\alpha}_0 = E_a I_a (M E_0)^{-1} \alpha_0 \tag{1.10}$$

Thus, the problem formulated reduces to determining that shape of the cross section $S_0(x)$ for which the volume V is minimal and the constraints (1.8) and (1.10) are satisfied.

2. Optimal shape of the cross-section.

Let us assume that the shape of the section is symmetric relative to the axes y and z, and its boundary is given in the first quadrant by a single-valued positive function $f(y)$, i.e. $z = f(y)$ for $z \geqslant 0$, $y \geqslant 0$. Let β_a denote the least root of the equation

$$F(\beta, \rho) = \bar{\alpha}_a; \quad \bar{\alpha}_a = \alpha_a I_a [M(x) z_a]^{-1}, \quad 0 \leqslant \beta \leqslant 1 \tag{2.1}$$

If (2.1) has no roots for $0 \leqslant \beta \leqslant 1$, then we put $\beta_a = 1$. The inequality (1.8) is equivalent to the following

$$\beta \leqslant \beta_a \tag{2.2}$$

It is seen from the definition of β and from (1.10) and (2.2) that the initial problem reduces to determining the function $f(y)$ (each for each section x) possessing properties noted, for which

$$\begin{gathered} S_0(x) \to \min, \; I_0(x) \geqslant C_1 \\ S_0(x) = 4\int_0^{y_0} f(y)\,dy, \quad I_0(x) = \frac{4}{3}\int_0^{y_0} f^3(y)\,dy \\ C_1 = \frac{I_a E_a}{E_0}\left(\max\left(\frac{1}{\beta_a}, \frac{r(x)}{\bar{\alpha}_0}\right) - 1\right) \end{gathered} \tag{2.3}$$

In a number of cases the optimal shape of the section is a rectangle. For instance, let a beam not be reinforced and let there be no lower bounds on the shape of the section. We assume that the function $f_0(y)$, $0 \leqslant y \leqslant y_0$ solves the problem (2.3) in a certain section x, and we consider in addition, a rectangle with sides parallel to the y, z axes which is symmetric relative to these axes. The altitude of this rectangle is $2r(x)$ and the width is $2l$, where l is determined from the condition $l = 3I_0(x)(4r^3(x))^{-1}$. The moments of intertia relative to the neutral axis of the rectangle and tne figure determined by the function $f_0(y)$ here are equal. Because of the condition $f_0(y) \leqslant r(x)$, the area of the rectangle $4r(x)l$ does not exceed $S_0(x)$ defined by the function $f_0(y)$. Consequently, in the case considered the optimal shape of the section is a rectangle. Even in the general case we shall see the optimal shape of the section among rectangles. It is here natural to consider that the main material in each section of the reinforcement is enclosed by the main material, i.e.,

$$z \geqslant z_a, \; y \geqslant y_a \tag{2.4}$$

This means that the problem (2.3) reduces to minimizing the product yz under the constraints (2.4) and the condition $4yz^3 \geqslant 3C_1$, where C_1 is defined in (2.3). We denote the altitude of the optimal rectangle by $2z_0$, and the width by $2y_0$. It is convenient to introduce the new variable $s = yz$. The formulated problem takes the following form in the variables s, z

$$\begin{gathered} s \to \min, \; s \geqslant \max[s_1(z), s_2(z), zy_a], \; z \geqslant z_a \\ s_1(z) = \frac{3}{4}\frac{E_a}{E_0} I_a (z\bar{\alpha}_0^{-1} - 1) z^{-2} \\ s_2(z) = \frac{3}{4}\frac{E_a}{E_0} I_a (\beta_a^{-1} - 1) z^{-2} \end{gathered} \tag{2.5}$$

Let us note that the optimal value $s_0 = y_0 z_0$ also satisfies the equation

$$s_0 = \min_{z \geqslant z_a} \max[s_1(z), s_2(z), zy_a] \tag{2.6}$$

Let us introduce the numbers $h_1, \ldots, h_5$ whose meaning is clear from Fig.2. The root h_1 is here the intersection of the curves s_1 and s_2, which always exists and is unique. The unique root h_2 also always exists. It can be verified that

$$\begin{gathered} h_1 = \bar{\alpha}_0 \beta_a^{-1}, \quad h_2 = \left[\frac{3}{4}\frac{a}{E_0 y_a} I_a (\beta_a^{-1} - 1)\right]^{1/3} \\ h_5 = 2\bar{\alpha}_0 = 2\alpha_0 E_a I_a (M E_0)^{-1} \end{gathered} \tag{2.7}$$

Finally, h_3 and h_4 are two positive roots of the cubic equation $s_1(z) = y_a z$; they may coincide, and can also not exist. Investigation of (2.5) and (2.6) shows that the following variants are possible depending on the parameters of the problem:

1) Let either $h_2 \leqslant h_3 \leqslant z_a \leqslant h_4$ or $h_3 \leqslant h_2 \leqslant h_4$ and $h_1 \leqslant z_a$. Then

$$z_0 = z_a,\ y_0 = s_1(z_a)/z_a \quad \text{as} \quad s_1(z_a) \leqslant y_a h_4$$
$$z_0 = h_4,\ y_0 = y_a \quad \text{as} \quad s_1(z_a) > y_a h_4$$

2) Let $h_3 \leqslant h_2 \leqslant h_4$ and $h_1 \geqslant z_a$. Then

$$z_0 = h_1,\ y_0 = s_2(h_1)/h_1 \quad \text{as} \quad s_2(h_1) \leqslant y_a h_4$$
$$z_0 = h_4,\ y_0 = y_a \quad \text{as} \quad s_2(h_1) > y_a h_4$$

In all the remaining cases (for instance, if $h_2 < h_3$ and $z_0 < h_3$, if $h_2 > h_4$, etc.) there will be

$$z_0 = \max(z_a, h_2),\ y_0 = y_a$$

Therefore, the optimal shape of the main material is a rectangle which can have either the minimal possible width or minimal altitude depending on the parameters of the problem.

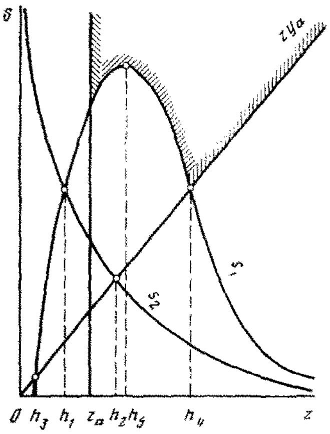

Fig.2

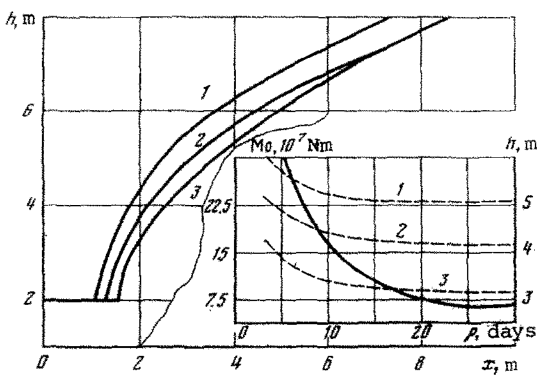

Fig.3

3. Optimal section shape for large values of the moment $M(x)$.

Let us find formulas for the numbers h_i as $M \to \infty$. Because of (2.7), we have

$$h_5 \to 0 \quad \text{as} \quad M \to \infty \tag{3.1}$$

Let us study the behavior of h_1. To do this, because of (2.7) it is sufficient to study the behavior of β_a as $M \to \infty$. The number β_a is defined as the root of (2.1) in which the quantity $\bar{\alpha}_a \to 0$ as $M \to \infty$. This means that by using the Lagrange inversion theorem (/5/, p.507), we obtain

$$\beta_a = \sum_{j=1}^{\infty} \gamma_j \bar{\alpha}_a^{\,j}, \quad \gamma_1 = [1 + J(0, \rho)]^{-1} = [1 + E_0 \varphi(\rho(x))]^{-1} > 0$$
$$\gamma_j = \frac{1}{j!} \frac{d^{j-1}}{d\beta^{j-1}} \left[\frac{1}{1 + (1-\beta) J(\beta, \rho)}\right]^j \Bigg|_{\beta=0}$$

It hence follows that

$$\frac{1}{\gamma_1 \bar{\alpha}_a} - \frac{1}{\beta_a} = \frac{1}{\gamma_1} \left[\sum_{j=1}^{\infty} \gamma_{j+1} \bar{\alpha}_a^{\,j}\right] \left[\sum_{j=1}^{\infty} \gamma_j \bar{\alpha}_a^{\,j}\right]^{-1}$$

Convergence of the series in the right side means that

$$\frac{1}{\beta_a} = \frac{1}{\gamma_1 \bar{\alpha}_a} + O(1) = \frac{1 + E_0 \varphi(\rho(x))}{\alpha_a I_a} z_a M + O(1), \quad M \to \infty$$

There results hence and from (2.7) and (1.10) that

$$h_1 \sim \frac{\alpha_0 E_a}{\alpha_a E_0}[1 + E_0\varphi(\rho(x))], \ z_a, \quad M \to \infty \tag{3.2}$$

$$h_2 \sim \left[\frac{3E_a I_a}{4E_0 y_a}\left(\frac{1+E_0\varphi(\rho(x))}{\alpha_a I_a}\right)M\right]^{1/3}, \quad M \to \infty$$

Finally, for sufficiently large M two positive roots h_3 and h_4 exist for the cubic equation $s_1(z) = y_a z$. The greatest of these h_4 satisfies the following relationship because of the Cardano formula

$$h_4 \sim \frac{z_a}{3}\left[\frac{3E_0}{4y_a z_a^2 E_a \alpha_0} M\right]^{1/3}, \quad M \to \infty \tag{3.3}$$

The relationships (3.2), (3.3) and (2.7) permit setting up the optimal section shape for sufficiently large values of the moment M. Let us present it. Because of (3.1) and (3.2), for sufficiently large M we will have

$$h_5 < h_1 < h_2 \tag{3.4}$$

Let us introduce the function

$$s_3(z) = \max(s_1(z), s_2(z)) = \begin{cases} s_2(z), & z \leqslant h_1 \\ s_1(z), & z \geqslant h_1 \end{cases}$$

We recall that $s_2(z)$ decreases monotonically, while $s_1(z)$ decreases monotonically for $z \geqslant h_5$. Because of (3.4), the function $s_3(z)$ therefore also decreases monotonically. On the basis of (2.6) we consequently conclude that the optimal value is $s_0 = \max(y_a h_2, y_a h_4)$. However, $h_2 \leqslant h_4$ (in the opposite case, there would be $h_1 > h_2$ because of the decrease in the function $s_2(z)$, which contradicts (3.4)). Hence, for sufficiently large M the optimal value is $s_0 = h_4 y_a$. This means that for large M

$$z_0 = h_4, \ y_0 = y_a \tag{3.5}$$

The dependence of the optimal altitude z_0 on the moment M is here given by (3.3). It is seen from the relationship (3.5) that for sufficiently large moments M the optimal shape is independent of the material age. However, that critical value of the moment M_0, starting with which the optimal section shape is given by (3.5), is already dependent on the material age, where M_0 diminishes as the age increases.

A graph of the dependence of M_0 on ρ is constructed numerically in Fig.3 for an aging function of the form

$$\varphi(\rho) = a_0/(\gamma\rho) + C_0 \tag{3.6}$$

for the following values of the parameters $I_a = 5.2$ m^4, $z_a = 2$ m, $y_a = 2$ m, $C_0 = 1.0\cdot 10^{-4}$ MPa^{-1}, $a_0 = 1.42\cdot 10^{-5}$ MPa, $\gamma = 0.03$ days^{-1}, $\alpha_a = 1.6\cdot 10^7$ N/m^2, $\alpha_0 = 1.5\cdot 10^6$ N/m^2, $E_a = 1.96\cdot 10^5$ MPa, $E_0 = 2.54\cdot 10^4$ MPa.

The optimal shape of a cantilever reinforced beam loaded by a lumped force at the free end has also been obtained numerically (the solid curves *1, 2, 3* correspond to values of ρ equal to 3,5, and 20 days). The dashed curves *1, 2, 3* depict the dependence of the altitude of the optimal section on the age ρ for values 12.10^7, 9.10^7, and 6.10^7 Nm of the moment, respectively.

We note also that

$$\partial\beta_a/\partial\rho \geqslant 0 \tag{3.7}$$

Hence, and from (2.6) there results that the dependence of the optimal transverse section shape on the age $\rho(x)$ will decrease as ρ increases for a fixed value of the moment $M(x)$. Moreover, the area of the optimal section will also decrease as ρ grows.

Let us prove the formula (3.7). If $\bar{\alpha}_a \geqslant 1$, then $\beta_a = 1$, , i.e., $\partial\beta_a/\partial\rho = 0$. This means that for $\bar{\alpha}_a \geqslant 1$ the inequality (3.7) is satisfied. Now, let $\bar{\alpha}_a < 1$. Then $\beta_a < 1$ and the following inequality is valid

$$\partial J(\beta, \rho)/\partial\rho < 0 \tag{3.8}$$

Indeed, by differentiating and integrating by parts the expression (1.8) for $J(\beta, \rho)$ we obtain

$$\partial J(\beta, \rho)/\partial\rho = \gamma J(\beta, \rho)\,[\varphi'(\rho)/\gamma\varphi(\rho) + \beta E_0\varphi(\rho) + 1] - \gamma E_0\varphi(\rho), \ \varphi' = \partial\varphi/\partial\rho$$

Hence, and from the positivity of the integral in (1.8) it is seen that

$$\operatorname{sign} \partial J(\beta,\rho)/\partial\rho = \operatorname{sign}\left[\beta+\psi(\rho)-\frac{1}{J(\beta,\rho)}\right] \tag{3.9}$$

$$\psi(\rho)=\frac{1}{E_0\varphi(\rho)}\left(1+\frac{\varphi'(\rho)}{\gamma\varphi(\rho)}\right)$$

It follows from the properties of the function $\varphi(\rho)$ that $1/\varphi(\rho)$ and $\varphi'(\rho)$ are monotonically increasing functions, while the limit of $\varphi'(\rho)$ equals zero as $\rho\to\infty$. Consequently, $\Psi(p)$ is a monotonically increasing function and

$$\lim_{\rho\to\infty}\psi(\rho)=(E_0C_0)^{-1} \tag{3.10}$$

If $\beta+\psi(\rho)\leqslant 0$, then (3.8) is satisfied by virtue of (3.9). Now, let $\beta+\psi(\rho)>0$. We rewrite (3.9) in the following equivalent form:

$$\operatorname{sign}\partial J(\beta,\rho)/\partial\rho=\operatorname{sign}[J(\beta,\rho)-J_1(\beta,\rho)] \tag{3.11}$$

$$J_1(\beta,\rho)=(\beta+\psi(\rho))^{-1}$$

Let us fix β. We assume that there exists such an age ρ_0 that $J(\beta,\rho_0)\geqslant J_1(\beta,\rho_0)$. We prove that for any $\rho>\rho_0$

$$J(\beta,\rho)>J_1(\beta,\rho_0) \tag{3.12}$$

Let us assume the opposite. Let ρ_1 be the first age greater than ρ_0 for which $J(\beta,\rho_1)\leqslant J_1(\beta,\rho_0)$. We examine the segment $[\rho_0,\rho_1]$. Since $J(\beta,\rho_0)\geqslant J(\beta,\rho_1)$, then because of the differentiability of the function (1.8) with respect to ρ, there exists a $\rho_2\in(\rho_0,\rho_1)$ such that $\partial J(\beta,\rho_2)/\partial\rho\leqslant 0$. Then on the basis of (3.11) there will be $J(\beta,\rho_2)\leqslant J_1(\beta,\rho_2)$. From the properties of $\psi(\rho)$ there results that $J_1(\beta,\rho)$ is continuous and decreases monotonically in ρ. Hence, it follows $J(\beta,\rho_2)<J_1(\beta,\rho_0)$ from the preceding inequality which contradicts the definition of ρ_1 since $\rho_2<\rho_1$. Therefore, the assumption about the existence of ρ_1 is false. The relationship (3.12) is proved.

On the basis of (3.10) we have

$$\lim_{\rho\to\infty}J_1(\beta,\rho)=q,\quad q=\left(\beta+\frac{1}{E_0C_0}\right)^{-1}$$

At the same time, from (1.8) there results the inequality

$$J(\beta,\rho)<\varphi(\rho)q/C_0$$

This means that for any $\varepsilon>0$ there exists a $\rho_3>\rho_0$ such that for $\rho>\rho_3$ there will be $J(\beta,\rho)<q+\varepsilon$. Setting $\varepsilon=J_1(\beta,\rho_0)-q>0$, we obtain

$$J(\beta,\rho)<q+(J_1(\beta,\rho_0)-q)=J_1(\beta,\rho_0),\quad\rho>\rho_3$$

This inequality contradicts (3.12). Hence, the assumption about the existence of ρ_0 is false. This means that for all ρ there will be $J(\beta,\rho)<J_1(\beta,\rho)$. Hence, (3.8) follows from (3.11).

Now, let us consider the solution β_a of (2.1) as a function of the age, i.e., $\beta_a=\beta_a(\rho)$. We write the equality

$$\frac{d}{d\rho}F(\beta_a(\rho),\rho)=\frac{\partial}{\partial\beta}F(\beta,\rho)\Big|_{\beta=\beta_a(\rho)}\cdot\frac{\partial\beta_a(\rho)}{\partial\rho}+\frac{\partial}{\partial\rho}F(\beta,\rho)\Big|_{\beta=\beta_a(\rho)}$$

By the definition of β_a we have $F(\beta_a(\rho),\rho)=\bar{\alpha}_a$ from (2.1), i.e.,

$$\frac{d}{d\rho}F(\beta_a(\rho),\rho)=0;\quad\frac{\partial}{\partial\beta}F(\beta,\rho)\Big|_{\beta=\beta_a}\geqslant 0$$

Taking (1.8) into account, we have $(\partial/\partial\rho)F(\beta,\rho)<0$ because of (3.8). Hence, $\partial\beta_a(\rho)/\partial\rho\geqslant 0$ is necessary to achieve the equality written above. Formula (3.7) is set up. From the preceding there results that $J(\beta,\rho)<J_1(\beta,\rho)$ and

$$\lim_{\rho\to\infty}J(\beta,\rho)=\lim_{\rho\to\infty}J_1(\beta,\rho)=q$$

We hence obtain the estimate

$$q<J(\beta,\rho)<[\beta+\psi(\rho)]^{-1}$$

These estimates are the more exact, the greater the ρ. Estimates for β_a follow from these inequalities and (2.1):

$$\bar{\alpha}_a\left[\frac{1-\bar{\alpha}_a}{\psi(\rho)}+1\right]^{-1}<\beta_a<\bar{\alpha}_a[(1-\bar{\alpha}_a)E_0C_0+1]^{-1}$$

which are convenient for large ρ as well as for large moments when $\bar{\alpha}_a$ is small.

For an aging function of the form (3.6) we have from (1.8)

$$J(\beta,\rho) = E_0(a_0+\rho C_0)\int_0^\infty \exp\left[-\gamma s\rho(1+\beta_1)\right](s+1)^{-\beta_1}\,ds$$

$$\beta_1 = \beta E_0 C_0$$

The integral in this formula is a degenerate Schlömilch hypergeometric function. In this case the inequality (3.7) can be set up directly by integrating the expression for $J(\beta,\rho)$ by parts.

The authors are grateful to N. Kh. Arutiunian for discussing the results obtained.

REFERENCES

1. ARUTIUNIAN N.Kh., Certain problems of creep theory for inhomogeneously aging bodies. Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No.3, (See also, in English, Some Problems in the Theory of Creep, Pergamon Press, Book No. 11507, 1966).
2. ARUTIUNIAN N.Kh. and KOLMANOVSKII V.B., Optimization problem in creep theory for inhomogeneous beams subjected to aging. Prikl. Mekhan., Vol.15, No.10, 1979.
3. ARUTIUNIAN N.Kh. and KOLMANOVSKII V.B., On the stability of nonuniformly aging viscoelastic rods. PMM, Vol.43, No.4, 1979.
4. RABOTNOV Iu.N., Elements of the Hereditary Mechanics of Solids. NAUKA, Moscow, 1977.
5. FIKHTENGOL'TS G.M., Differential and Integral Calculus Course, Vol.2, FIZMATGIZ, Moscow, 1959.

Translated by M.D.F.